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ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF A DYNAMICAL PROBLEM FOR AN ELASTIC HALF-SPACE: THE AXISYMMETRIC CASE†

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Simple analytic expressions are derived for the stresses in an elastic half-space when the load is applied at times near the starting time. Asymptotic expansions of the stresses as $t \to 0$ are developed and error bounds worked out for the principal terms of the asymptotic series, subject to certain assumptions. It is assumed, in particular, that the Hankel transform of the radial distribution function of the load decreases at an exponential rate.

THE RESULTS reported here complement some other constructions [1–3] of asymptotic approximations to the solution of this particular problem of the dynamic theory of elasticity. In particular, an earlier analysis of the axially symmetric version of the problem [3] focused on the construction and investigation of solutions for sources of perturbations of the delta-function type for the coordinates and unit jumps or delta-functions for the time. The forms of the solution were quite complicated. The technique employed in [3] yielded comparatively simple expressions for the displacements only in the long-time range, as well as at the boundary of the half-space and on the axis of symmetry.

1. In cylindrical coordinates r, φ, z consider an elastic half-space $z \geqslant 0$, at rest prior to the time $t=0$. Beginning at the time $t=0$, a normal axially symmetric load

$$T_z(r,t) = T_0 f(r) a(t) \tag{1.1}$$

where T_0 is a constant, $f(r)$ admits of a Hankel transformation and $a(t)$ is the source function, is applied at the free boundary of the half-space. It is required to determine the stresses the half-space.

We will change to non-dimensional variables

$$r' = r/\delta, \quad z' = z/\delta, \quad t' = c_2 t/\delta \tag{1.2}$$

where c_2 is the velocity of propagation of transverse elastic waves and δ is some characteristic dimension. Henceforth, the primes will be omitted.

The potentials of the elastic displacements $\Phi(r, z, t)$, $\Psi(r, z, t)$ are defined as solutions of certain boundary-value problems for the wave equations

$$\Delta\Phi - \gamma^2 \frac{\partial^2\Phi}{\partial t^2} = 0, \quad \left(\Delta - \frac{1}{r^2}\right)\Psi - \frac{\partial^2\Psi}{\partial t^2} = 0$$

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$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},\quad \gamma^2=\frac{c_2^2}{c_1^2}=\frac{1-2\nu}{2(1-\nu)} \tag{1.3}$$

$$\Phi\big|_{t=0}=\frac{\partial\Phi}{\partial t}\bigg|_{t=0}=\Psi\big|_{t=0}=\frac{\partial\Psi}{\partial t}\bigg|_{t=0}=0,\quad |\Phi|<\infty, |\Psi|<\infty$$

(c_1 is the velocity of propagation of longitudinal elastic waves and ν is Poisson's ratio).

The solutions of problems (1.3) are determined by using the Laplace and Hankel transformations of zeroth and first orders. The transforms of the unknown potentials will be

$$\Phi^*(r,z,s)=L_s\{\Phi(r,z,t)\}=\int_0^\infty \lambda C(\lambda,s)e^{-zR_1}J_0(\lambda r)d\lambda$$

$$\Psi^*(r,z,s)=\int_0^\infty \lambda D(\lambda,s)e^{-zR_2}J_1(\lambda r)d\lambda \tag{1.4}$$

$$R_1=\sqrt{\gamma^2 s^2+\lambda^2},\quad R_2=\sqrt{s^2+\lambda^2};\quad \arg R_1=\arg R_2=0 \text{ for } s>0$$

where L_s is the Laplace transform and J_0 and J_1 are Bessel functions of the first kind.

Having determined the transforms of the stresses corresponding to Φ^* and Ψ^*, we then find the unknown functions $C(\lambda,s)$, $D(\lambda,s)$ from the boundary conditions

$$\sigma_{zz}^*\big|_{z=0}=-T_0a^*(s)\int_0^\infty \lambda f^H(\lambda)J_0(\lambda r)d\lambda,\quad \sigma_{rz}^*\big|_{z=0}=0$$

$$f^H(\lambda)=\int_0^\infty rf(r)J_0(\lambda r)dr,\quad a^*(s)=L_s\{a(t)\}$$

The final formulae for the transforms of the stresses are

$$\frac{\sigma_{jj}^*}{T_0}=\int_0^\infty \lambda^3 f^H(\lambda)U_j(V_1-V_2)d\lambda-\frac{k_j}{2}\int_0^\infty \lambda f^H(\lambda)J_0(\lambda r)s^2V_1d\lambda,\quad j=r,\varphi,z$$

$$\frac{\sigma_{rz}^*}{T_0}=\int_0^\infty \lambda^2 f^H(\lambda)J_1(\lambda r)\left(\frac{R^2V_2}{R_2}-R_1V_1\right)d\lambda,\ R^2=\frac{s^2}{2}+\lambda^2,\ P(\lambda,s)=R^4-\lambda^2R_1R_2 \tag{1.5}$$

$$U_\varphi=J_1(\lambda r)/(\lambda r), U_z=-J_0(\lambda r), U_r=-U_\varphi-U_z, k_r=k_\varphi=\nu/(1-\nu), k_z=1$$

$$V_1=a^*(s)R^2e^{-zR_1}/P(\lambda,s),\quad V_2=a^*(s)R_1R_2e^{-zR_2}/P(\lambda,s)$$

Formal expressions for the inverse transforms, as contour integrals, may now be derived formally from the inversion theorem. However, this solution is of little practical value.

2. To obtain asymptotic formulae for the exact solution, valid as $t\to 0$, we will confine ourselves to functions $f(r)$ and $a(t)$ satisfying the following additional conditions. We shall assume that $f^H(\lambda)$ is an exponentially decreasing function of λ. In particular, this implies the convergence of the integrals

$$A_{m,n}(r)=\int_0^\infty \lambda^m f^H(\lambda)J_n(\lambda r)d\lambda;\quad m=1,2,\ldots;\quad n=0,1. \tag{2.1}$$

Note that this condition is satisfied by the dome-shaped distributions $f(r)=e^{-r^2/4}$, $f(r)=(r^2+1)^{-(2n+1)/2}$, $n=0, 1, \ldots$, which are frequently used in the theory of elasticity.

We shall also assume that the function $a^*(s)$ can be represented in the neighbourhood of $s=\infty$ by an absolutely convergent series

$$a^*(s)=\sum_{n=0}^{\infty}\frac{\xi_n}{s^{\mu_n}},\quad 0<\mu_0<\mu_1<\ldots,\quad \lim_{n\to\infty}\mu_n=\infty \tag{2.2}$$

We will demonstrate the method by deriving an asymptotic expansion for σ_{zz}. After some simple reduction, introducing obvious notation, we write

$$\frac{\sigma^*_{zz}}{T_0}=-\sigma^{*(1)}_{zz}-\sigma^{*(2)}_{zz}+\sigma^{*(3)}_{zz} \tag{2.3}$$

$$\sigma^{*(k)}_{zz}e^{\gamma_k zs}=\int_0^{\infty}\lambda f^H(\lambda)J_0(\lambda r)F_k(\lambda,z,s)d\lambda,\quad k=1,2,3$$

$$F_1(\lambda,z,s)=a^*(s)e^{-z(R_1-\gamma s)},\quad F_k(\lambda,z,s)=a^*(s)\frac{\lambda^2R_1R_2e^{-z(R_{k-1}-\gamma_k s)}}{P(\lambda,s)},k=2,3$$

$$\gamma_1=\gamma_2=\gamma,\quad \gamma_3=1$$

Further manipulations give rise to the representation

$$F_1(\lambda,z,s)=a^*(s)\exp\left[-\frac{z\lambda^2}{\gamma s\left(1+\sqrt{1+\lambda^2/(\gamma^2s^2)}\right)}\right]$$

from which it follows that the series

$$F_1(\lambda,z,s)=\sum_{n=0}^{\infty}\varphi^*_n(z,s)\lambda^{2n} \tag{2.4}$$

is convergent for any fixed value of λ and $|s|>M_1$.

The functions $\varphi^*_n(z,s)$ can be represented in the neighbourhood of $s=\infty$ by generalized power series. By the theorem on the expansion of transforms in generalized power series [4], we have

$$\begin{aligned}&\varphi_n(z,t)=L_t^{-1}\{\varphi^*_n(z,s)\}=L_t^{-1}\left\{\frac{d_n(z)}{s^{n+\mu_0}}+\ldots\right\}=\frac{d_n(z)}{\Gamma(n+\mu_0)}t^{n+\mu_0-1}+\ldots\\&\varphi_{n+1}(z,t)=o(\varphi_n(z,t)),\quad t\to 0,\quad n=0,1,\ldots\end{aligned} \tag{2.5}$$

where L_t^{-1} is the inverse of the operator L_s.

Substituting (2.4) into (2.3) and integrating term by term with respect to λ, we obtain a formal expansion

$$\sigma^{*(1)}_{zz}e^{\gamma zs}\approx\sum_{n=0}^{\infty}A_{2n+1,0}(r)\varphi^*_n(z,s)$$

Transforming back to original functions and using the delay theorem, we obtain

$$\sigma^{(1)}_{zz}\approx\sum_{n=0}^{\infty}A_{2n+1,0}(r)\varphi_n(z,t-\gamma z),\quad t\to 0,\quad t>\gamma z \tag{2.6}$$

It can be proved that (2.6) is indeed an asymptotic expansion of $\sigma^{(1)}_{zz}$ as $t\to 0$.

The first condition of an asymptotic expansion is satisfied by (2.5).

Let us estimate the order of magnitude as $t\to 0$ of the difference $r_n(r,z,t)$ between the left-hand side and the first $n+1$ terms on the right of Eq. (2.6) By (2.1) and (2.3) we have

$$r_n(r,z,t+\gamma z)=\int_0^{\infty}\lambda f^H(\lambda)J_0(\lambda r)L_t^{-1}\{q^*_n(\lambda,z,s)\}d\lambda \tag{2.7}$$

$$q_n^*(\lambda,z,s) = F_1(\lambda,z,s) - \sum_{k=0}^{n} \varphi_k^*(z,s)\lambda^{2k}$$

justifying the passage to original functions under the integral sign by the uniform convergence of the integrals, which in turn follows from the assumed exponential rate of decrease of $f^H(\lambda)$.

By (2.4) and (2.5) we obtain the following expansion, valid for $|s|>M_1$

$$q_n^*(\lambda,z,s) = b_{n+1}(\lambda,z)/s^{n+\mu_0+1}+\ldots$$

using which, we finally obtain

$$r_n(r,z,t) = o(\varphi_n(z,t-\gamma z)), \quad t\to 0, \quad \gamma z<t \tag{2.8}$$

so that the second condition of an asymptotic expansion also holds.

An asymptotic expansion for $\sigma_{zz}^{(2)}$ is obtained in the same way as for $\sigma_{zz}^{(1)}$, since the only roots of the equation $P(\lambda, s)=0$ are $s=0$, $s=\pm i\lambda\vartheta$, $0<\vartheta<1$ [2] and therefore, for any constant λ and $|s|>M_2$, the function $F_2(\lambda, z, s)$ has the same properties as $F_1(\lambda, z, s)$. Similar arguments yield asymptotic expansions for $\sigma_{zz}^{(3)}$ and the other stresses.

Note that, as follows from the derivation of (2.6), the latter holds for any z and t such that $t-\gamma z\to +0$.

We also note that the functions $\varphi_n^*(z, s)$ in (2.4) and other similar expansions, $n\geqslant 1$, are products of $a^*(s)$ and regular rational fractions of s; hence their inversion presents no difficulties.

The asymptotic expansions of the stresses, retaining terms of order $1+\mu_0$ as $t\to 0$, are as follows:

$$\begin{gathered}
\frac{\sigma_{jj}}{T_0} \approx -k_jA_{10}a(t-\gamma z)+k_jA_{30}\left[\frac{z}{2\gamma}a_1(t-\gamma z)+2(1-2\gamma)a_2(t-\gamma z)\right]- \\
-w_j[4\gamma a_2(t-z)-2a_2(t-\gamma z)]+\ldots, \quad j=r,\varphi,z \\
\frac{\sigma_{rz}}{T_0} \approx -2\gamma A_{21}[a_1(t-\gamma z)-a_1(t-z)]+\ldots \\
w_\varphi = A_{21}/r, \quad w_z=-A_{30}, \quad w_r=-w_\varphi-w_z \\
a_{k+1}(t)=\int_0^t a_k(\tau)d\tau, \quad a_0(t)=a(t), \quad k=0,1,\ldots
\end{gathered} \tag{2.9}$$

We recall that the functions $a_k(t)$ in (2.9) are inverse transforms and therefore vanish for negative values of the arguments.

3. Retaining a finite number of terms in (2.9), we obtain an approximate solution of the problem. The use of such approximations in practical work is legitimate if accompanied by an error bound.

Let us estimate the error in the asymptotic representation of the solution. Separating out the principal terms of the asymptotic expansions in (2.9), we obtain

$$\begin{gathered}
\sigma_{jj} = -k_jf(r)a(t-\gamma z)+\delta_{jj}, \quad j=r,\varphi,z \\
\sigma_{rz} = -2\gamma A_{21}(r)[a_1(t-\gamma z)-a_1(t-z)]+\delta_{rz}
\end{gathered} \tag{3.1}$$

These relations are exact expressions for the stresses. An approximate solution is found by dropping the errors δ_{jj}, δ_{rz} in (3.1) We note that the normal stresses in the approximate solution are the product of $f(r)$ and the solution of the corresponding one-dimensional problem.

We will first estimate δ_{zz}. From (2.3) and (3.1), introducing the obvious notation, we obtain

$$\delta_{zz}=\delta_{zz}^{(1)}-\delta_{zz}^{(2)}=L_t^{-1}\left\{\int_0^\infty \lambda f^H(\lambda)J_0(\lambda r)a^*(s)\varphi^*(\lambda,z,s)d\lambda\right\}-$$

$$-L_t^{-1}\left\{\int_0^\infty \lambda^3 f^H(\lambda)J_0(\lambda r)a^*(s)\psi^*(\lambda,z,s)d\lambda\right\} \tag{3.2}$$

$$\varphi^*(\lambda,z,s)=e^{-\gamma zs}-e^{-zR_1},\quad \psi^*(\lambda,z,s)=R_1R_2(e^{-zR_1}-e^{-zR_2})/P(\lambda,s)$$

By [5]

$$L_t^{-1}\{\varphi^*(\lambda,z,s)\}=\frac{\lambda^2 z}{\gamma}\frac{J_1(\lambda\gamma^{-1}\sqrt{t^2-\gamma^2z^2})}{\lambda\gamma^{-1}\sqrt{t^2-\gamma^2z^2}}\eta(t-\gamma z)$$

where $\eta(x)$ is the Heaviside unit function. Using the convolution theorem and the inequalities $|J_1(x)/x|\leqslant \tfrac{1}{2}$, $|J_0(x)|\leqslant 1$ [6] we find that

$$|\delta_{zz}^{(1)}|\leqslant\frac{z}{2\gamma}\beta_3 b_1(t-\gamma z),\quad \beta_k=\int_0^\infty\lambda^k|f^H(\lambda)|d\lambda,\quad b_1(t)=\int_0^t|a(\tau)|d\tau \tag{3.3}$$

To estimate $\delta_{zz}^{(2)}$, we use the inversion theorem. We first estimate

$$\psi(\lambda,z,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\psi^*(\lambda,z,s)e^{st}ds=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(\lambda,z,s,t)ds$$

$$F(\lambda,z,s,t)=\frac{R_1R_2(e^{-zR_1}\eta_1-e^{-zR_2}\eta_2)}{P(\lambda,s)}e^{st},\quad \eta_1=\eta(t-\gamma z),\ \eta_2=\eta(t-z) \tag{3.4}$$

The insertion of η_1 and η_2 does not alter the value of the integral in (3.4), since the integrals of functions with factors $\exp(-zR_1)$ and $\exp(-zR_2)$ vanish when $t<\gamma z$ and $t<z$, respectively.

Slit the complex s-plane along the intervals $(-i\infty,\ -i\lambda]$, $[i\lambda,\ i\infty)$, and let C be the contour formed by the straight line $\operatorname{Re}s=\sigma$, arcs of the circle $|s|=R_0$, the sides of the cuts $[iR_0,\ -i(\lambda\gamma^{-1}+\rho_0)]$, $[-i(\lambda\gamma^{-1}+\rho_0),\ -i(\lambda+\rho_0)]$, $[i(\lambda+\rho_0),\ i(\lambda\gamma^{-1}-\rho_0)]$, $[i(\lambda\gamma^{-1}+\rho_0),\ iR_0]$ and the circles $|s\pm i\lambda|=\rho_0$, $|s\pm i\lambda\gamma^{-1}|=\rho_0$. Applying Cauchy's residue theorem to the integral of $F(\lambda,\ z,\ s,\ t)$ along C, letting $R_0\to\infty$, $\rho_0\to 0$ and using Jordan's lemma, we obtain

$$\psi(\lambda,z,t)=\Sigma+I_\gamma$$

where Σ is the sum of residues of the function at the singular points $s=0$, $s=\pm i\lambda\vartheta$, and I_γ is the sum of the integrals along the sides of the cuts.

For Σ, using the inequalities

$$|\sin x|\leqslant|x|,\ |\exp(-\lambda z\sqrt{1-\gamma^2\vartheta^2})-\exp(-\lambda z\sqrt{1-\vartheta^2})|\leqslant\lambda z$$

(which are proved using Taylor's formula), we have

$$|\Sigma|\leqslant\frac{2t}{1-\gamma^2}\eta_{12}+8\alpha t(\eta_{12}+\lambda z\eta_2) \tag{3.5}$$

$$\alpha=\frac{(1-\vartheta^2)(1-\gamma^2\vartheta^2)}{\vartheta^6-6\vartheta^4+(12-8\gamma^2)\vartheta^2-4(1-\gamma^2)},\quad \eta_{12}=\eta_1-\eta_2$$

We will show that the number α in (3.5) is bounded. To verify this, we have to determine the domain of possible ϑ values. Solving the identity $P(\lambda, i\lambda\vartheta)=0$ for γ^2 and using the conditions $0<\vartheta^2<1$, $0<\gamma^2\leqslant 0.5$, we obtain a system of inequalities $8(\vartheta^2-1)\leqslant\vartheta^6-8\vartheta^4+24\vartheta^4-16<0$, whose solution yields

$$3-\sqrt{5}=0.7639...\leqslant\vartheta^2<0.9126... \tag{3.6}$$

Methods of analysis can be used to prove that α is bounded in the set (3.6)
To evaluate I_γ, we set $s=i\lambda y$ and introduce obvious notation, to obtain

$$\frac{\lambda\pi}{8}I_\gamma=\sum_{k=1}^{5}I_k=-\eta_{12}\int_1^{1/\gamma}\rho_1\rho_3\rho^4e^{-\lambda\rho_1 z}\frac{\sin\lambda yt}{P_1(y)}dy+$$
$$+\eta_2\int_1^{1/\gamma}4\rho_1^2\rho_3^2\sin\lambda\rho_3 z\frac{\sin\lambda yt}{P_1(y)}dy-\eta_2\int_1^{1/\gamma}\rho_1\rho_3\rho^4(e^{-\lambda\rho_1 z}-\cos\lambda\rho_3 z)\frac{\sin\lambda yt}{P_1(y)}dy-$$
$$-\eta_{12}\int_{1/\gamma}^{\infty}\rho_2\rho_3\sin\lambda\rho_2 z\frac{\sin\lambda yt}{P_2(y)}dy-\eta_2\int_{1/\gamma}^{\infty}\rho_2\rho_3(\sin\lambda\rho_2 z-\sin\lambda\rho_3 z)\frac{\sin\lambda yt}{P_2(y)}dy$$
$$\rho_1=\sqrt{1-\gamma^2y^2},\quad \rho_2=\sqrt{\gamma^2y^2-1},\quad \rho_3=\sqrt{y^2-1},\quad \rho^2=1-\rho_3^2$$
$$P_1(y)=\rho^8+16\rho_1^2\rho_3^2,\quad P_2(y)=\rho^4+4\rho_1\rho_2$$

The values of the radical are determined from the conditions in (1.4) and the continuity of the arguments, as done in [3].
Using the relations

$$P_1(y)\geqslant 8\rho^4\rho_1\rho_3,\quad P_1(y)\geqslant 16\rho_1^2\rho_3^2,\quad |e^{-\lambda\rho_1 z}-\cos\lambda\rho_3 z|\leqslant\lambda z(1+\gamma^{-1})$$

we get

$$|I_1|\leqslant\frac{\lambda t}{16}\frac{1-\gamma^2}{\gamma^2}\eta_{12},\quad |I_2|\leqslant\frac{\lambda^2 zt}{12}\frac{1-\gamma^3}{\gamma^3}\eta_2,\quad |I_3|\leqslant\frac{\lambda^2 zt}{16}\frac{(1-\gamma^2)(1+\gamma)}{\gamma^3}\eta_2 \tag{3.7}$$

We will now estimate I_4. Changing the variable of integration by the substitution $x=\lambda z\sqrt{(\gamma^2y^2-1)}$, using the additivity of integrals and estimating the integrand, we obtain

$$|I_4|\leqslant\lambda\gamma^2 z\eta_2\left[\int_0^h|\chi(x)|dx+\int_h^\infty|\chi(x)|dx\right],\quad 0<h<\infty \tag{3.8}$$
$$|\chi(x)|\leqslant t\gamma^{-1}z^{-1},\quad 0\leqslant x\leqslant h;\quad |\chi(x)|\leqslant x^{-2},\quad h\leqslant x<\infty$$

This inequality implies an estimate for I_4 which holds for any h.
Minimizing this estimate as a function of h and doing the same for I_5, we find

$$|I_4|\leqslant 2\gamma\lambda t\eta_{12},\quad |I_5|\leqslant 2\lambda\gamma\sqrt{zt}\eta_2 \tag{3.9}$$

Using estimates (3.5), (3.7) and (3.9) and proceeding as for (3.3), we obtain

$$|\delta_{zz}^{(2)}|\leqslant\beta_3 z\left(\frac{2}{1-\gamma^2}+8\alpha+\frac{1-\gamma^2}{2\pi\gamma^2}+\frac{2\gamma}{\pi}\right)[b_1(t-\gamma z)-b_1(t-z)]+$$
$$+\beta_4 zt\left[8\alpha+\frac{(1-\gamma^2)(1+\gamma)}{2\pi\gamma^3}+\frac{2(1-\gamma^3)}{3\pi\gamma^3}\right]b_1(t-z)+\beta_3\sqrt{zt}\frac{16\gamma}{\pi}b_1(t-z) \tag{3.10}$$

Estimates of δ_{rr} and $\delta_{\varphi\varphi}$ are derived by similar means.

To derive an estimate for δ_{rz}, one proceeds as follows. It follows from (1.5) and (3.1), after suitable reduction, that

$$\delta_{rz} = \delta_{rz}^{(1)} - \delta_{rz}^{(2)} = \int_0^\infty \lambda^4 f^H(\lambda) J_1(\lambda r) L_t^{-1}\left\{\frac{a^*(s)}{R_1} g_1^*(\lambda, z, s)\right\} d\lambda -$$

$$-2\gamma \int_0^\infty \lambda^2 f^H(\lambda) J_1(\lambda r) L_t^{-1}\{a^*(s) g_2^*(\lambda, z, s)\} d\lambda$$

$$g_1^*(\lambda, z, s) = \left[(1-2\gamma^2)\frac{R^2}{R_1 R_2} + 2\gamma^2\right]\psi^*(\lambda, z, s)$$

$$g_2^*(\lambda, z, s) = \frac{e^{-\gamma z s} - e^{-zs}}{s} - \gamma \frac{e^{-zR_1} - e^{-zR_2}}{R_1}$$

A bound for $\delta_{rz}^{(1)}$ is obtained by the same method as for $\delta_{zz}^{(2)}$, using the relation $L_t^{-1}\{R_1^{-1}\} = \gamma^{-1} J_0(\lambda \gamma^{-1} t)$ [5]. To estimate $\delta_{rz}^{(2)}$ one uses the representation

$$g_2(\lambda, z, t) = L_t^{-1}\left\{\frac{e^{-\gamma z s}}{s} - \gamma \frac{e^{-zR_1}}{R_1}\right\}\eta_{12} + L_t^{-1}\left\{\gamma \frac{e^{-zR_2} - e^{-zR_1}}{R_1}\right\}\eta_2$$

the known inverses of the transforms $e^{-\gamma z s}/s$, e^{-zR_1}/R_1, the theorem on the correspondence of the original functions for $g(s)$ and $g(\sqrt{(s^2+\lambda^2)})$ [5], as well as the convolution theorem.

Combining our results, we obtain

$$|\delta_{jj}| \leq \beta_3\left\{k_j\left[\alpha_3 t + \frac{16\gamma\sqrt{\gamma}}{\pi}\sqrt{zt}\right] + \frac{z}{2\gamma}[k_j(1+2\alpha_2) + \zeta_j \alpha_2]\right\} b_1(t-\gamma z) +$$

$$+\left\{\frac{\beta_3}{2}\left[\alpha_3 t + \frac{16\gamma\sqrt{zt}}{\pi}\right] + \beta_4 \frac{1-\gamma^3}{\gamma^3}\frac{zt}{3\pi}\right\}\zeta_j b_1(t-z), \quad j = r, \varphi.$$

$$|\delta_{zz}| \leq |\delta_{zz}^{(1)}| + |\delta_{zz}^{(2)}|$$

$$|\delta_{rz}| \leq \frac{r}{2}\left\{\beta_5 z\left(\frac{\gamma_1^2 \alpha_2}{\gamma} + 2\gamma\alpha_3 + \frac{4\gamma^2}{\pi}\right)[b_2(t-\gamma z) - b_2(t-z)] + \right.$$

$$+2\beta_5 z^2 \frac{1-\gamma}{\gamma}[b_1(t-\gamma z) - b_1(t-z)] + \beta_6 zt\left(\frac{16\gamma_1^2\alpha}{\gamma(2-\vartheta^2)} + \right. \tag{3.11}$$

$$+\frac{\gamma_1^2}{\pi\gamma^3}\left[\left(1+\frac{1}{\gamma}\right)(\gamma_1 + \alpha_1) + \frac{1}{6\gamma^2}(4\gamma_1^3 + \gamma_2^3 - 3\gamma_1^2\gamma_2)\right] + 2\gamma_1^2(2-\gamma^2) +$$

$$\left.+2\gamma\left[8\alpha + \frac{\gamma_2^2(1+\gamma)}{2\pi\gamma^3} + \frac{2(1-\gamma^3)}{3\pi\gamma^3}\right]\right) b_2(t-z) + 8\beta_5\sqrt{zt}\left(\frac{\gamma_1^2\gamma_2\sqrt{2}}{\pi\gamma} + \frac{4\gamma^2}{\pi}\right) b_2(t-z) +$$

$$\left.+\beta_5 zt\left(2\gamma + \gamma_2^2 \frac{1+\sqrt{2}}{\gamma}\right) b_1(t-z)\right\}$$

where $|\delta_{zz}^{(i)}|$ $(i = 1, 2)$ were estimated in (3.3) and (3.10) and we have used the following notation

$$\zeta_r = 1+\sqrt{2},\quad \zeta_\varphi = 1,\quad \gamma_1 = \sqrt{1-2\gamma^2},\quad \gamma_2 = \sqrt{1-\gamma^2},\quad \alpha_1 = \frac{3\gamma^2-1}{2\gamma}\arcsin\frac{3\gamma^2-1}{\gamma_2^2}$$

$$\alpha_2 = 2/\gamma_2^2 + 16\alpha/(2-\vartheta^2) + \gamma_1/(\pi\gamma^2) + \alpha_1/(\pi\gamma) + 8/\pi$$

$$\alpha_3 = 2/\gamma_2^2 + 8\alpha + \gamma_2^2/(2\pi\gamma^2),\quad b_2(t) = \int_0^t b_1(\tau)d\tau$$

4. Consider the following example. Suppose that, in non-dimensional variables

$$f(r) = (r^2+1)^{-3/2},\quad a(t) = \eta(t)$$

Then $f^H(\lambda) = e^{-\lambda}$; the coefficients A_{mn} are expressed in terms of elementary functions [7], $\beta_k = k!$, $a_k = b_k = t^k/k!$. Putting $\nu = 0.25$, $r = z = 0.01$, we see that for $t = 0.5(1-\gamma)z = 7.887\times 10^{-3}$

$$\sigma_{rr}/T_0 = -0.3333 \pm 3.0\times 10^{-3},\quad \sigma_{\varphi\varphi}/T_0 = -0.3333 \pm 1.8\times 10^{-3}$$

$$\sigma_{zz}/T_0 = -0.9998 \pm 1.1\times 10^{-3},\quad \sigma_{rz}/T_0 = 7.318\times 10^{-5} \pm 4.6\times 10^{-7}$$

and the maximum relative error is 0.9%; if $t = 1.1z = 1.1\times 10^{-2}$, the maximum relative error is 2.4%.

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